

Chapter 4 Online Appendix: The Mathematics of Utility Functions

We saw in the text that utility functions and indifference curves are different ways to represent a consumer's preferences. Calculus can help us make the connection between the two curves more explicit. In fact, in this appendix we see that we can derive indifference curves directly from a consumer's utility function.

It's easiest to show this derivation by beginning with a specific utility function. Let's look at a consumer with the utility function $U(X,Y) = 3XY$, where X and Y are two goods. Any consumer's indifference curve shows all the bundles of goods that generate the same level of utility. For our consumer, let's say that utility level is 300 utils: $U(X,Y) = 3XY = 300$.

What does this formula tell us? It is precisely the consumer's indifference curve at 300 utils. We could stop here, but graphing a function is usually easier if we rewrite it with Y as a function of X . So, first we solve for Y :

$$3XY = 300$$
$$Y = \frac{300}{3X} = \frac{100}{X}$$

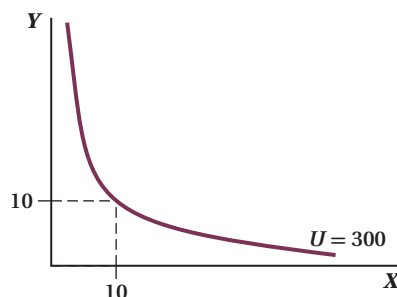
Figure 4OA.1 graphs this function to display the consumer's indifference curve at 300 utils.

The slope of this indifference curve decreases as the consumer trades off Y for X , or as she consumes fewer units of Y and more of X . In other words, as she moves down her indifference curve, the consumer has to receive more of X as she gives up units of Y in order to stay on the same indifference curve.

We can confirm this relationship by calculating the marginal rate of substitution (MRS) from the utility function using calculus. First, find the marginal utilities by taking the partial derivative of the utility function with respect to each of the goods:

$$MU_X = \frac{\partial U(X,Y)}{\partial X} = 3Y$$
$$MU_Y = \frac{\partial U(X,Y)}{\partial Y} = 3X$$

Figure 4OA.1 An Indifference Curve at 300 Utils



Then use the marginal utilities to calculate the MRS :

$$MRS_{XY} = \frac{MU_X}{MU_Y} = \frac{3Y}{3X} = \frac{Y}{X}$$

Now that we have the marginal rate of substitution, we can find out what happens to the relative consumption of X and Y as we move along the consumer's indifference curve. To see what occurs as X increases, we take the partial derivative of MRS_{XY} with respect to X :

$$\frac{\partial MRS_{XY}}{\partial X} = \frac{\partial(X^{-1}Y)}{\partial X} = -X^{-2}Y < 0$$

Just as the graph demonstrates, the consumer's marginal rate of substitution—the slope of the indifference curve—decreases as her consumption of X increases along the indifference curve. In other words, when the consumer has more units of X , she is willing to give up fewer units of Y to get an additional unit of X ; while keeping her utility level constant.

40A.1 figure it out

Suppose that a consumer has the following utility function:

$$U(X, Y) = 5X^{0.5} + Y$$

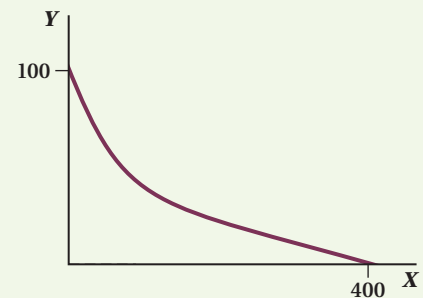
- Graph the indifference curve generated by the utility function when $\bar{U} = 100$.
- Find the marginal rate of substitution and relate the shape of the indifference curve to it.

Solution:

- First, set the utility function equal to 100 units and solve for Y as a function of X :

$$\begin{aligned} 5X^{0.5} + Y &= 100 \\ Y &= 100 - 5X^{0.5} \end{aligned}$$

Notice that either X or Y may equal zero; therefore, the indifference curve intersects both axes. The vertical intercept is 100 units of Y , and the horizontal intercept is 400 units of X .



- The graph indicates that the slope of the indifference curve decreases as the consumer gives up Y to get more X . Therefore, the MRS is decreasing. Let's confirm this by calculating the marginal rate of substitution. First, find the marginal utilities:

$$MU_X = \frac{\partial U(X, Y)}{\partial X} = \frac{\partial(5X^{0.5} + Y)}{\partial X} = 2.5X^{-0.5}$$

$$MU_Y = \frac{\partial U(X, Y)}{\partial Y} = \frac{\partial(5X^{0.5} + Y)}{\partial Y} = 1$$

Now find the marginal rate of substitution:

$$MRS_{XY} = \frac{MU_X}{MU_Y} = 2.5X^{-0.5}$$

Finally, take the partial derivative of this marginal rate of substitution with respect to X to see how the MRS is altered with changes in X :

$$\frac{\partial MRS_{XY}}{\partial X} = \frac{\partial(2.5X^{-0.5})}{\partial X} = (-0.5)2.5X^{-1.5} = -1.25X^{-1.5} < 0$$

Therefore, the MRS decreases as the consumption of X increases along the indifference curve; as shown in the figure.

Monotonic Transformations of Utility Functions

A utility function is a convenient way to represent preferences. However, we saw in the chapter that utility functions have many limitations. One limitation is that although utility functions tell us a lot about the ordinal rankings of goods, they reveal nothing about cardinal rankings. In other words, we know what a consumer prefers, but not the strength of that preference. Because of this, we can shift, stretch, or squeeze the utility function into any form as long as we don't change the ordering of preferences over bundles, and we'll still have the same representation of the relative utility of goods. This type of transformation is termed a *monotonic transformation*. Monotonic transformations tell us a great deal about the properties of utility functions, and so it's worth going into more detail about these types of transformations here.

Some monotonic transformations are simple. Adding or subtracting a fixed number of "utils" from a utility function doesn't change the ordering of the preferences. It just makes the utility level shift up or down by some constant number of units. Multiplying a utility function by any positive number is also simple. This change multiplies the number of units of utility by that positive number without changing their order. The same goes for exponents. As long as we apply the same positive exponent to the whole utility function, we just increase or decrease the number of utils by that exponent without changing the order.¹ Let's try to make this more concrete with a specific example.

Consider the utility function $U(X,Y) = X^{0.5}Y^{0.5}$, and the following two bundles: bundle A has 4 units of X and 16 units of Y (4, 16), and bundle B has 9 units of X and 4 units of Y (9, 4). The consumer prefers bundle A to bundle B because bundle A generates 8 units of utility and B generates 6 units of utility. Now let's consider some monotonic transformations of $U(X,Y)$. Suppose we multiply the utility function by 37 to get $U_2(X,Y) = 37X^{0.5}Y^{0.5}$. The consumer still prefers A with 296 units of utility to B with 222 units of utility. What if we square the original utility function to get $U_3(X,Y) = (X^{0.5}Y^{0.5})^2 = XY$? Then she still prefers A to B (64 versus 36 units of utility). In each of these transformations, the only thing that changes is the number of units of utility.

Checking the ordering of many bundles by plugging them into the transformed utility functions is impractical. Therefore, we need another way to show us whether the transformation preserves the preference ordering.

Let's think about these transformations in terms of indifference curves. The utility

¹ These factors and exponents need to be positive. If you multiply a utility function by a negative number or raise it to a negative exponent, you will invert the preference ordering.

function determines the shape of the indifference curve and assigns a number to the level of utility represented by each indifference curve. We have already discussed how the number assigned to the level of utility doesn't matter. As long as the number assigned to the level of utility increases as we move away from the origin, the preference ranking is preserved.

What does matter is the shape of the indifference curve. If a transformation makes the indifference curves steeper or flatter, then the ordering of bundles on the new indifference curves will be very different from the original ordering of those bundles, and the transformation cannot be monotonic.

How do we know if the shape of the indifference curves is preserved? By looking—once again—at the marginal rate of substitution. Recall that the *MRS* is the negative of the slope of the indifference curve at any bundle and is equal to the ratio of the marginal utilities at that point. If the marginal rate of substitution at any point doesn't change, we know that we have a monotonic transformation.

Let's go back to our example from above, where $U(X,Y) = X^{0.5}Y^{0.5}$. For this utility function,

$$MRS_{XY} = \frac{MU_X}{MU_Y} = \frac{\frac{\partial X^{0.5}Y^{0.5}}{\partial X}}{\frac{\partial X^{0.5}Y^{0.5}}{\partial Y}} = \frac{0.5X^{-0.5}Y^{0.5}}{0.5X^{0.5}Y^{-0.5}} = \frac{Y}{X}$$

What if we transform the utility function by squaring and then taking the natural log? Then,

$$U_2(X,Y) = \ln[(X^{0.5}Y^{0.5})^2] = \ln(XY) = \ln X + \ln Y$$

This certainly looks very different from our original function. In fact, though, we can solve for the *MRS* to see that it is a monotonic transformation (recall that the derivative of $\ln(Z)$ with respect to Z is $1/Z$):

$$MRS_{XY} = \frac{MU_X}{MU_Y} = \frac{\frac{\partial(\ln X + \ln Y)}{\partial X}}{\frac{\partial(\ln X + \ln Y)}{\partial Y}} = \frac{\frac{1}{X}}{\frac{1}{Y}} = \frac{Y}{X}$$

The equal marginal rates of substitution tell us that the shapes of the indifference curves are the same for $U(X,Y)$ and $U_2(X,Y)$. Therefore, $U_2(X,Y)$ is a monotonic transformation of $U(X,Y)$, and the two utility functions represent the same set of preferences.

Let's take the lessons learned from these examples and see if we can demonstrate a general principle of monotonic transformations of Cobb–Douglas utility functions, the form of the original utility function we presented here. For a generic Cobb–Douglas utility function

$$U(X,Y) = AX^\alpha Y^\beta$$

the marginal rate of substitution is

$$MRS_{XY} = \frac{MU_X}{MU_Y} = \frac{\frac{\partial AX^\alpha Y^\beta}{\partial X}}{\frac{\partial AX^\alpha Y^\beta}{\partial Y}} = \frac{\alpha AX^{\alpha-1}Y^\beta}{\beta AX^\alpha Y^{\beta-1}} = \frac{\alpha Y}{\beta X}$$

When will a transformation be monotonic for a Cobb–Douglas utility function? Examining the fraction above, we can see that any transformation of the Cobb–Douglas utility function will be a monotonic transformation as long as the ratio of α to β does not change. The magnitude of the exponents doesn't matter. The only thing that matters is the relative size of the exponents. For example, multiplying both exponents by any real number would not change the ratio of α to β . Therefore, we could raise the utility function to any power without changing the preference ordering.

40A.2 figure it out

Royce only cares about how much coffee he drinks. He can buy coffee in 10-ounce cups or in 20-ounce cups. Royce's preferences are represented by the following utility function:

$$U_1(X, Y) = X + 2Y$$

where X is the number of 10-ounce cups and Y is the number of 20-ounce cups. Determine whether or not Royce's preferences could also be represented by this utility function:

$$U_2(X, Y) = (50X + 100Y)^{0.5}$$

Solution:

Notice that Royce views these goods as perfect substitutes and that he is twice as happy with a 20-ounce cup of coffee as he is with a 10-ounce cup of coffee. Take the partial derivatives to get the marginal utilities:

$$MU_X = \frac{\partial U_1(X, Y)}{\partial X} = \frac{\partial(X + 2Y)}{\partial X} = 1$$

$$MU_Y = \frac{\partial U_1(X, Y)}{\partial Y} = \frac{\partial(X + 2Y)}{\partial Y} = 2$$

So, the marginal utility of each good is constant, and an additional unit of Y is twice as good as an additional unit of X . Consequently, the marginal rate of substitution is

$$MRS_{XY} = \frac{MU_X}{MU_Y} = \frac{1}{2}$$

indicating that Royce is always willing to trade $\frac{1}{2}$ of a 20-ounce cup for another 10-ounce cup.

We can determine whether the second utility function is a monotonic transformation of the original utility function by comparing the marginal rates of substitution. First, find the marginal utilities:

$$MU_X = \frac{\partial U_2(X, Y)}{\partial X} = (0.5)50(50X + 100Y)^{-0.5} = 25(50X + 100Y)^{-0.5}$$

$$MU_Y = \frac{\partial U_2(X, Y)}{\partial Y} = (0.5)100(50X + 100Y)^{-0.5} = 50(50X + 100Y)^{-0.5}$$

Notice that the marginal utilities for the transformed utility function are very different from the original marginal utilities. However, marginal utilities are not the deciding factor in determining whether Royce's preferences can be represented by the second utility function; the equality of the marginal rates of substitution tells us whether the second utility function is a monotonic transformation of the first.

We now may find the marginal rate of substitution:

$$MRS_{XY} = \frac{MU_X}{MU_Y} = \frac{25(50X + 100Y)^{-0.5}}{50(50X + 100Y)^{-0.5}} = \frac{1}{2}$$

which is exactly the same as the marginal rate of substitution for the original utility function. Therefore, the utility functions are monotonic transformations of one another and represent the same set of preferences even though the numbers we assign to the utility levels are very different.

Corner Solutions

A solution to the utility-maximization problem is an “interior” solution if the consumption bundle contains at least some of each good, because such a bundle will be located in the interior of the X - Y plane. Almost all the utility-maximization problems we have seen in the chapter have interior solutions. There is another possibility, however: A consumer may be better off consuming just one good and not consuming any of the other. Economists call this a “corner solution,” and we’ll consider these solutions to the utility-maximization problem here.

Let’s consider, however, why we more commonly see interior solutions. Consider what is probably the most commonly seen utility function in economics: the Cobb–Douglas utility function, $U(X,Y) = AX^\alpha Y^\beta$. Utility here is only maximized when the consumption of both goods is positive and nonzero. It’s fairly easy to see why. If a consumer consumes none of one good (say, she consumes 0 units of X), then $U(X = 0, Y) = A(0)^\alpha Y^\beta = 0$. As a result, utility is never maximized at a corner solution. This is true of more than just Cobb–Douglas functions: Any time you are working with a utility function that gives zero utility whenever one good is not consumed, you will have an interior solution.

To have a corner solution, the utility function must result in positive utility when the consumer only consumes one good. In the chapter, we saw one such set of indifference curves where corner solutions are possible in Figure 4.22. In that illustration, Greg is a consumer of romance novels R and economics textbooks T . Let’s say that Greg faces the following utility function:

$$U(R,T) = RT + 20T$$

where R is the quantity of romance novels and T is the quantity of economics textbooks. A corner solution is possible with this utility function, but it is of a particular type. Greg may consume all economics texts and no romance novels; then his utility is

$$U(0,T) = 0T + 20T = 20T$$

Therefore, a corner solution is possible. If Greg consumes 0 economics textbooks, however, his utility goes to 0. As a result, we know that Greg’s utility-maximizing bundle may have 0 romance novels, but must have textbooks.

Let’s solve a utility-maximization problem to illustrate this. Suppose that Greg faces the budget constraint drawn in Figure 4.22: He has \$240 to spend on the goods, a romance novel costs \$20, and the price of an economics textbook is \$120. In this case, Greg solves the following constrained optimization problem:

$$\max_{R,T} U(R,T) = RT + 20T \text{ s.t. } 240 = 20R + 120T$$

We can write his problem as a Lagrangian and take the first-order conditions:

$$\max_{R,T,\lambda} \mathcal{L}(R,T,\lambda) = RT + 20T + \lambda(240 - 20R - 120T)$$

FOC:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial R} &= T - 20\lambda = 0 \\ \frac{\partial \mathcal{L}}{\partial T} &= R + 20 - 120\lambda = 0 \\ \frac{\partial \mathcal{L}}{\partial \lambda} &= 240 - 20R - 120T = 0 \end{aligned}$$

Solve for λ in the first two FOCs:

$$\begin{aligned} T - 20\lambda &= 0 \\ T &= 20\lambda \\ \lambda &= \frac{1}{20}T \end{aligned}$$

and

$$\begin{aligned}
 R + 20 - 120\lambda &= 0 \\
 R + 20 &= 120\lambda \\
 \lambda &= \frac{1}{120}(R + 20)
 \end{aligned}$$

Set these expressions for λ equal to each other and solve for T as a function of R :

$$\begin{aligned}
 \frac{1}{20}T &= \frac{1}{120}(R + 20) \\
 T &= \frac{1}{6}(R + 20)
 \end{aligned}$$

Now plug this expression for T into the third FOC:

$$\begin{aligned}
 240 - 20R - 120T &= 240 - 20R - 120\left[\frac{1}{6}(R + 20)\right] = 0 \\
 240 - 20R - 20R - 400 &= 0 \\
 40R &= -160 \\
 R &= -4
 \end{aligned}$$

So, Greg’s optimal number of romance novels is -4 . That is impossible, so Greg will do the next best thing. He will choose one of the corners of his budget constraint—a corner solution—where he consumes 0 romance novels and 2 economics textbooks. Note that this bundle does not satisfy our utility-maximization condition in which the marginal rate of substitution is equal to the ratio of the prices (or where the indifference curve and budget constraint are tangent), but it’s as close as Greg can get to his utility-maximizing bundle under these conditions.

Sometimes choosing one corner of the budget constraint is obvious, as in the case where the consumer wants a negative number of one of the goods. But there’s another type of problem that will yield a corner solution. We’ll see this in the Figure It Out below.

40A.3 figure it out

Recall from the previous Figure It Out that Royce has the utility function

$$U(X, Y) = X + 2Y$$

where X is the number of 10-ounce cups of coffee and Y is the number of 20-ounce cups of coffee. Suppose Royce has \$6 to spend on coffee each day and that the price of a 10-ounce cup is \$2 and the price of a 20-ounce cup is \$3. Find Royce’s optimal bundle of 10-ounce and 20-ounce cups of coffee.

Solution:

We could write out Royce’s constrained optimization problem and solve the Lagrangian, but let’s instead apply the utility-maximization condition we derived in the chapter and in the appendix:

$$MRS_{XY} = \frac{MU_X}{MU_Y} = \frac{P_X}{P_Y}$$

In this problem,

$$MRS_{XY} = \frac{1}{2}$$

but

$$\frac{P_X}{P_Y} = \frac{2}{3}$$

so there is no way that the utility-maximization condition will be met. Royce will either want to consume all 10-ounce cups or all 20-ounce cups. The easiest way to determine which he will choose is to calculate Royce’s utility at each corner of his budget constraint. If he only buys 10-ounce cups, he will get 3 cups and 3 units of utility. On the other hand, he could buy two 20-ounce cups and get 4 units of utility. He is better off with the 20-ounce cups so that will be the corner solution to his problem.

Problems

- For the following utility functions,
 - find the marginal rate of substitution,
 - discuss how MRS_{XY} changes as the consumer substitutes X for Y along an indifference curve,
 - derive the equation for the indifference curve where utility is equal to a value of 100, and
 - graph the indifference curve where utility is equal to a value of 100:
 - $U(X,Y) = XY^2$
 - $U(X,Y) = XY + 10Y$
 - $U(X,Y) = X^2 + Y^2$
- Suppose that Renée consumes 2 goods, X and Y , and that her preferences are represented by the utility function $U(X,Y) = X^{0.5}Y^{0.5}$. Determine whether each of the following utility functions also represent Renée's preferences:
 - $V(X,Y) = 33XY + 427$
 - $W(X,Y) = 400X^{0.75}Y^{0.25}$
- Suppose that a consumer has a choice between gallon jugs of milk and half-gallon jugs of milk. The consumer only cares about how much milk she consumes and does not care about the size of the jug.
 - Write a utility function for this consumer.
 - Suppose the price of a half-gallon jug of milk is \$2, the price of a gallon jug of milk is \$5, and the consumer has \$10 to spend on milk. Set up a Lagrangian and derive the first-order conditions for the maximization problem.
 - Is there a solution to the first-order conditions? If so, solve for the optimal quantities of gallon jugs and half-gallon jugs. If not, what is the solution to the maximization problem?
- Terry has \$100 to spend on lake perch and walleye for his family's Friday fish fry. Terry's utility function is

$$U(P,W) = 10W + PW$$
 where P is pounds of lake perch and W is pounds of walleye.
 - Suppose the price of perch is \$5 per pound and the price of walleye is \$3 per pound. Find Terry's optimal combination of perch and walleye.
 - Suppose that the price of perch increases to \$12 per pound, while the price of walleye remains unchanged. Find Terry's optimal combination of perch and walleye.