# Chapter 5 Online Appendix:
The Calculus of Demand

The chapter uses the graphical model of consumer choice to derive the demand for a good by changing the price of the good and the slope of the budget constraint and then plotting the relationship between the prices and the optimal quantities as the demand curve. Figure 5.7 shows this process for three different prices, three budget constraints, and three points on the demand curve.

You can see that this graphical approach works well if you need to plot a few points on a demand curve; however, getting the quantity demanded at every price requires us to apply the calculus of utility maximization. Before we do the calculus, let's review what we know about the demand curve. The chapter defines the demand curve as "the relationship between the quantity of a good that consumers demand and the good's price, holding all other factors constant." In the appendix to Chapter 4, we always

Figure 5.7   **Building an Individual's Demand Curve**
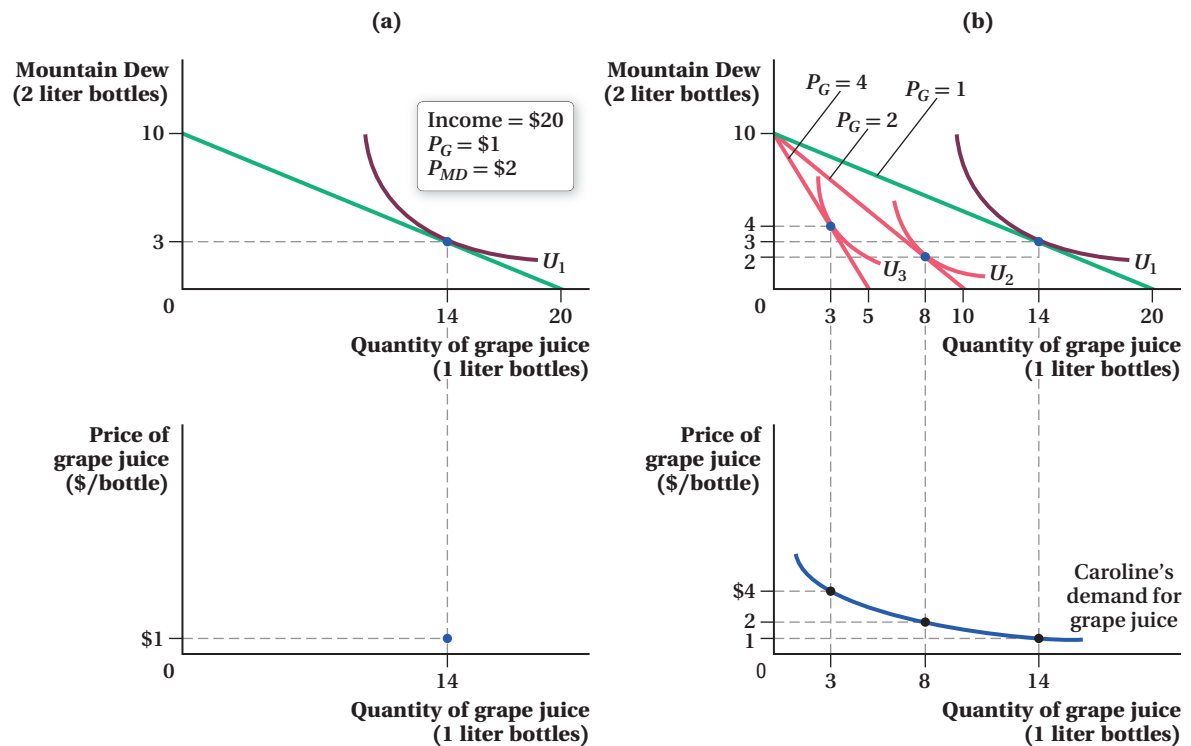
(a) At her optimal consumption bundle, Caroline purchases 14 bottles of grape juice when the price per bottle is $1 and her income is $20. The bottom panel plots this point on her demand curve, with the price of grape juice on the y-axis and the quantity of grape juice on the x-axis.

(b) A completed demand curve consists of many of these quantity-price points. Here, the optimal quantity of grape juice consumed is plotted for the prices $1, $2, and $4 per bottle. This creates Caroline's demand curve, as shown in the bottom panel.

solved the consumer's utility-maximization problem for a specific price, so the main change in deriving a demand curve is that the price of one good will be a variable instead of a constant. The other parts of our consumer choice model represent the "holding all other factors constant" part of the definition of the demand curve. That is, the consumer's income, the utility function, and the price of the other good will not change.

Now let's derive the demand curve. Suppose that a consumer only consumes two goods, X and Y, and has the familiar Cobb–Douglas utility function $U(X, Y) = X^{\alpha}Y^{1-\alpha}$, where $0 < \alpha < 1$. The consumer has a fixed income, $\overline{I}$, and the price of good Y is fixed at $\overline{P_Y}$. The price of good X is variable and equal to P_X. The consumer's utility-maximization problem is

$$\max_{X,Y} U(X, Y) = X^{\alpha}Y^{1-\alpha} \text{ s.t } \overline{I} = P_X X + \overline{P_Y}Y$$

We can rewrite this as a Lagrangian:

$$\max_{X,Y,\lambda} \mathcal{L} = X^{\alpha}Y^{1-\alpha} + \lambda(\overline{I} - P_X X - \overline{P_Y}Y)$$

Take the first-order conditions (FOCs):

$$\frac{\partial \mathcal{L}}{\partial X} = \alpha X^{\alpha-1}Y^{1-\alpha} - \lambda P_X = 0$$

$$\frac{\partial \mathcal{L}}{\partial Y} = (1 - \alpha)X^{\alpha}Y^{-\alpha} - \lambda\overline{P_Y} = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = \overline{I} - P_X X - \overline{P_Y}Y = 0$$

Solving for λ in the first two constraints gives us the necessary condition for utility maximization for this Cobb–Douglas utility function:

$$\lambda = \frac{\alpha X^{\alpha-1}Y^{1-\alpha}}{P_X} = \frac{(1 - \alpha)X^{\alpha}Y^{-\alpha}}{\overline{P_Y}}$$

To simplify, collect the price terms on the right-hand side and the other terms on the left-hand side:

$$\frac{\alpha}{(1 - \alpha)}X^{\alpha-1}X^{-\alpha}Y^{1-\alpha}Y^{\alpha} = \frac{P_X}{P_Y}$$

$$\frac{\alpha Y}{(1 - \alpha)X} = \frac{P_X}{P_Y}$$

Solve for Y as a function of X and substitute into the third first-order condition:

$$Y = \frac{(1 - \alpha)P_X}{\alpha\overline{P_Y}}X$$

$$\overline{I} - P_X X - \overline{P_Y}Y = 0$$

$$\overline{I} = P_X X + \overline{P_Y}\frac{(1 - \alpha)P_X}{\alpha\overline{P_Y}}X$$

$$\overline{I} = P_X X\left[1 + \frac{(1 - \alpha)}{\alpha}\right]$$

$$\overline{I} = P_X X\left[\frac{\alpha}{\alpha} + \frac{(1 - \alpha)}{\alpha}\right]$$

$$\overline{I} = \frac{P_X X}{\alpha}$$

$$X(P_X; \overline{P_Y}, \overline{I}) = \frac{\alpha\overline{I}}{P_X}$$

$$= \alpha\overline{I}P_X^{-1}$$

Thus, the demand for X is a function of the price of X holding the price of Y and income constant. We can confirm that this demand curve satisfies the Law of Demand by taking the partial derivative of the demand with respect to P_X:

$$\frac{\partial X(P_X; \overline{P_Y}, \overline{I})}{\partial P_X} = -\alpha \overline{I} P_X^{-2} < 0$$

As expected, the quantity of X demanded is inversely related to the price of X, all else equal. Notice that the demand for X is not affected by changes in the price of Y. We discussed this quirk of the Cobb–Douglas utility function in the Chapter 4 appendix.

We can also use the calculus of the consumer choice model to see how the demand curve shifts when something other than the good's own price changes. If there is a discrete change in income or the price of the other good, we change the constraint on the utility maximization problem and solve for the new demand curve. If tastes for a good change, we derive the new demand curve using the new utility function that represents the new tastes.

Finally, we can use the constrained maximization problem to derive other relationships by using the same approach we adopted to derive the demand curve. For example, the Engel curve shows the relationship between the quantity of a good consumed and a consumer's income, all else equal. To derive the Engle curve, we solve the consumer's utility-maximization problem with income as a variable while keeping the prices of the goods constant.

5OA.1 figure it **out**

Let's return to the problem of Antonio from the Figure It Out exercises in the calculus appendices for Chapters 4 and 5 in the text. Antonio gets utility from burgers (B) and fries (F). His utility function is

$$U(B,F) = \sqrt{BF} = B^{0.5}F^{0.5}$$

Antonio's income is $20, and the price of fries is $2.

a. Derive Antonio's demand for burgers.

b. Confirm that Antonio wants to purchase 2 burgers when the price of burgers is $5 each and 1 burger when the price of burgers is $10 each.

c. Suppose Antonio's income increases to $40. Derive Antonio's new demand curve. Are burgers a normal or inferior good for Antonio? Explain.

Solution:

a. To solve for Antonio's demand for burgers, we need to solve his constrained optimization problem:

$$\max_{B,F} B^{0.5}F^{0.5} \text{ s.t. } 20 = pB + 2F$$

where B is the quantity of burgers he eats, F is servings of fries, and p is the price of burgers. Let's apply the utility-maximization condition:

$$MRS_{BF} = \frac{MU_B}{MU_F} = \frac{P_B}{P_F}$$

$$\frac{MU_B}{MU_F} = \frac{0.5B^{-0.5}F^{0.5}}{0.5B^{0.5}F^{-0.5}} = \frac{F}{B} = \frac{p}{2} = \frac{P_B}{P_F}$$

Solve for F as a function of B and p:

$$\frac{F}{B} = \frac{p}{2}$$

$$F = 0.5pB$$

Plug this expression for F into the budget constraint and solve for B:

$$20 = pB + 2F = pB + 2(0.5pB) = 2pB$$

$$B = \frac{10}{p}$$

is the demand for burgers.

b. When the price of burgers is $5, Antonio wants to consume 2 burgers ($10/2$), and when the price of burgers is $10, Antonio wants to consume 1 burger ($10/10$). This confirms the results we obtained in the chapter and in the appendix.

c. Antonio's demand for burgers will change when his income increases from $20 to $40. His constrained optimization problem is now

$$\max_{B,F} B^{0.5}F^{0.5} \text{ s.t. } 40 = pB + 2F$$

The change in his income only changes his constraint. Therefore, the utility-maximization condition and the relationship between F, B and the price of burgers do not change. We can just plug this relationship into the new budget constraint and solve for B:

$$40 = pB + 2F = pB + 2(0.5pB) = 2pB$$
$$B = \frac{20}{p}$$

$B = \frac{20}{p}$ is his new demand. Notice that Antonio wants to consume more burgers at every price after his income increases. Therefore, burgers are a normal good for Antonio.

Hicksian Demand Curves

Economists call the demand curves that we just derived *Marshallian demand curves*. These demand curves show the relationship between price and quantity demanded, assuming that the consumer's income stays constant. We know from the chapter that the purchasing power of this fixed income changes when the good's price changes. That's why Marshallian demand curves have an income effect as well as a substitution effect. Sometimes we want demand curves that only have a substitution effect.[1] We call these "Hicksian demand curves" or "compensated demand curves."

To find the substitution effect of a price change, we looked at how the optimal consumption of a good changes while keeping utility constant. To isolate the substitution effect, we solved the consumer's expenditure-minimization problem given the new price and the consumer's original utility level. We'll use the same approach to derive the Hicksian demand curves.

Suppose that a consumer has utility for goods X and Y given by $U(X, Y) = X^\alpha Y^{1-\alpha}$, where $0 < \alpha < 1$, utility is fixed at $\overline{U}$, and the price of good Y is fixed at $\overline{P_Y}$. The price of good X is variable and equal to P_X. The consumer's expenditure-minimization problem is

$$\min_{X,Y} P_X X + \overline{P_Y} Y \text{ s.t. } \overline{U} = X^\alpha Y^{1-\alpha}$$

Rewrite the consumer's problem as a Lagrangian:

$$\min_{X,Y,\lambda} \mathcal{L} = P_X X + \overline{P_Y} Y + \lambda(\overline{U} - X^\alpha Y^{1-\alpha})$$

Take the first-order conditions (FOCs):

$$\frac{\partial \mathcal{L}}{\partial X} = P_X - \lambda \alpha X^{\alpha-1} Y^{1-\alpha} = 0$$

$$\frac{\partial \mathcal{L}}{\partial Y} = \overline{P_Y} - \lambda(1-\alpha) X^\alpha Y^{-\alpha} = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = \overline{U} - X^\alpha Y^{1-\alpha} = 0$$

Solve for λ in the first two constraints:

$$\lambda = \frac{P_X}{\alpha X^{\alpha-1} Y^{1-\alpha}} = \frac{\overline{P_Y}}{(1-\alpha) X^\alpha Y^{-\alpha}}$$

or

$$\frac{\alpha Y}{(1-\alpha) X} = \frac{P_X}{\overline{P_Y}}$$

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[1] For example, the deadweight loss from an excise tax is caused only by the substitution effect of the change in prices, not the income effect. Thus, if you use a Marshallian demand curve—which includes both the substitution and income effects—to measure deadweight loss, you'll overstate the efficiency cost of the tax.

Solve for Y as a function of X and substitute into the third first-order condition:

$$Y = \frac{(1-\alpha)P_X}{\alpha \overline{P_Y}}X$$

$$\overline{U} = X^\alpha Y^{1-\alpha} = X^\alpha \left[\frac{(1-\alpha)P_X}{\alpha \overline{P_Y}}X\right]^{1-\alpha} = X\left[\frac{(1-\alpha)P_X}{\alpha \overline{P_Y}}\right]^{1-\alpha}$$

Solving for X gives us the Hicksian demand:

$$X(P_X; \overline{P_Y}, \overline{U}) = \overline{U}\left[\frac{\alpha \overline{P_Y}}{(1-\alpha)P_X}\right]^{1-\alpha} = P_X^{-(1-\alpha)}\overline{U}\left[\frac{\alpha \overline{P_Y}}{(1-\alpha)}\right]^{1-\alpha}$$

Like Marshallian demand, the Hicksian demand function satisfies the Law of Demand. We can show this by taking the partial derivative of the Hicksian demand with respect to P_X:

$$\frac{\partial X(P_X; \overline{P_Y}, \overline{U})}{\partial P_X} = -(1-\alpha)\overline{U}\left[\frac{\alpha \overline{P_Y}}{(1-\alpha)}\right]^{1-\alpha} P_X^{\alpha-2} < 0$$

because $0 < \alpha < 1$.

We can also see that the quantity of X demanded would increase if the price of Y were to increase. In other words, X and Y are substitutes. This substitution effect may surprise you because we've shown several times that Cobb–Douglas utility functions have the property that the demand for each good is not a function of the price of the other good. But remember: The Hicksian demand doesn't give us the total effect of a change in price on the demand for another good; it just represents the substitution effect. In fact, for a Cobb–Douglas function, the substitution effect will be offset by a corresponding change in demand from the income effect. We can see this income effect by examining the Marshallian demand curve, which we solved for in the previous section:

$$X(P_X; \overline{P_Y}, \overline{I}) = \frac{\alpha \overline{I}}{P_X}$$

5OA.2 figure it out

Recall from our earlier example that Antonio gets utility from burgers (B) and fries (F). His utility function is

$$U(B,F) = \sqrt{BF} = B^{0.5}F^{0.5}$$

Suppose that the price of fries is \$2 and Antonio's utility is $10^{0.5}$. Derive Antonio's Hicksian demand for burgers.

Solution:

To get Antonio's Hicksian demand for burgers, we need to solve his constrained optimization problem:

$$\min_{B,F} pB + 2F \text{ s.t. } 10^{0.5} = B^{0.5}F^{0.5}$$

where p is the price of burgers. As we showed in the appendix to Chapter 4, the necessary condition for expenditure minimization is the same as that for utility maximization so we can use the result from the earlier example:

$$F = 0.5pB$$

Plug this expression for F into the utility constraint and solve for B:

$$10^{0.5} = B^{0.5}F^{0.5} = B^{0.5}(0.5pB)^{0.5} = B(0.5p)^{0.5}$$

$$B = \frac{10^{0.5}}{(0.5p)^{0.5}} = \left(\frac{10}{0.5}\right)^{0.5}p^{-0.5} \approx 4.47p^{-0.5}$$

Notice that the quantity of burgers demanded is inversely related to the price of burgers in the Hicksian demand, as expected.

Recall that Antonio wanted to consume 2 burgers when burgers cost \$5 each and he had \$20 of income. Under those conditions, his utility was $10^{0.5}$ units. The Hicksian demand tells us that Antonio wants to consume 2 burgers when the price of burgers is \$5 each and he has $10^{0.5}$ units of utility. So, the Marshallian and Hicksian demands give us the same consumption of burgers when burgers cost \$5 each as long as the utility level for the Hicksian demand is the maximum utility Antonio can achieve when his income is fixed at the level used for the Marshallian demand. We'll use this relationship to derive the Slutsky equation in the next section.

The Slutsky Equation

We showed in the chapter and the appendix that the total effect of a price change may be decomposed into the substitution and income effects. The Slutsky equation is a handy way of summarizing this decomposition, and it can be written in broad terms as

$$\text{Total effect} = \text{substitution effect} + \text{income effect}$$

Given our knowledge of Marshallian and Hicksian demands, we can derive the Slutsky equation more formally.

To begin, let's decompose the income and substitution effects of a change in the price of one good in the optimal consumption bundle. We found the initial consumption bundle (X^*, Y^*) by solving the consumer's utility-maximization problem given the consumer's income (I), the price of good X (P_X), and the price of good Y (P_Y). The initial consumption bundle yields some level of utility $(\overline{U})$. We could have found this same initial bundle using expenditure minimization instead of utility maximization. All we'd need to know is the level of utility $\overline{U}$ instead of the consumer's income I.

We can represent this initial bundle using both Marshallian and Hicksian demands. We derived Marshallian demand from utility maximization given the prices and income. So, the initial quantity of good X obtained from utility maximization could have been obtained by plugging the price of X into the Marshallian demand:

$$X^* = X(P_X, P_Y, I)$$

On the other hand, we derived the Hicksian demand from expenditure minimization given the prices and utility level, so we could have obtained the initial quantity of good X by plugging the price of X into the Hicksian demand:

$$X^* = H(P_X, P_Y, \overline{U})$$

Because utility maximization and expenditure minimization produce the same results in this instance, we know that the Marshallian and Hicksian demands have to yield the same initial quantity of X, given the prices of the goods, income, and the initial level of utility:

$$X^* = X(P_X, P_Y, I) = H(P_X, P_Y, \overline{U})$$

This is the crucial fact on which the Slutsky equation hinges. To find it, differentiate this equation with respect to P_X:

$$\frac{\partial X(P_X, P_Y, I)}{\partial P_X} + \frac{\partial X(P_X, P_Y, I)}{\partial I} \frac{dI}{dP_X} = \frac{\partial H(P_X, P_Y, \overline{U})}{\partial P_X}$$

Recognizing that $I = P_X X + P_Y Y$, we can simplify this further by substituting $\frac{dI}{dP_X} = X$:

$$\frac{\partial X(P_X, P_Y, I)}{\partial P_X} + \frac{\partial X(P_X, P_Y, I)}{\partial I} X = \frac{\partial H(P_X, P_Y, \overline{U})}{\partial P_X}$$

Rearrange the equation to find the Slutsky equation:

$$\frac{\partial X(P_X, P_Y, I)}{\partial P_X} = \frac{\partial H(P_X, P_Y, \overline{U})}{\partial P_X} - X \frac{\partial X(P_X, P_Y, I)}{\partial I}$$

This is exactly equal to what we sought to find—the equation that summarizes:

$$\text{Total effect} = \text{substitution effect} + \text{income effect}$$

In particular, the left-hand side is the total effect of the price change, or how the consumption of X changes as the price of X changes. The first term on the right-hand side is the substitution effect: how the consumption of X changes as the price of X changes, holding utility constant. Finally, the second term on the right-hand side shows the income effect. Notice that the income effect depends on both how much of

the good the consumer buys (X), as well as how much the quantity of X changes when purchasing power is transformed $\left(\dfrac{\partial X(P_X,P_Y,I)}{\partial I}\right)$.

We can use the Slutsky equation to examine the directions of the income and substitution effects and to think about why demand curves typically slope downward. First, the sign of the substitution effect is always negative: Consumers always consume less of a good when the price increases (or more when the price decreases), holding utility constant. However, the direction of the income effect depends on whether the good is normal or inferior and the size of the income effect depends on how much of the good is purchased. If the good is normal, the income effect will be positive; that is, an increase in income increases consumption or a decrease in income decreases consumption. In the Slutsky equation, the income effect is subtracted from the substitution effect; therefore, the income and substitution effects work in the same direction for a normal good, reducing the quantity of the good consumed when the price increases (or increasing the quantity consumed when the price decreases).

For an inferior good, quantity increases when purchasing power decreases and the income effect is negative. Subtracting this negative income effect offsets part of the negative substitution effect. In other words, the income effect works in the opposite direction of the substitution effect and the net outcome depends on the relative size of the two effects. Almost always, the substitution effect is larger than the income effect, and the sign on the total effect is negative; therefore, the demand curve slopes downward. However, for a Giffen good, the income effect dominates, and the demand curve slopes upward. The Slutsky equation shows us why Giffen goods are theoretically possible: The income effect just needs the correct sign and magnitude. It also shows why they are extremely rare: The consumer needs to buy a lot of an inferior good and consumption of the good needs to be very responsive to changes in income to make the income effect larger than the substitution effect and make the demand curve upward-sloping.

Problems

1. Suppose that there are two goods, X and Y. The utility function is $U = X^2Y$. The price of X is P, and the price of Y is \$2. Income is \$90. Derive the demand for X as a function of P.

2. Suppose that there are two goods, X and Y. The utility function is $U = XY + X$. The price of X is P, and the price of Y is \$40. Income is \$200. Derive the demand for X as a function of P.

3. Suppose that Maureen believes that name-brand laundry detergent is twice as effective as generic laundry detergent but that the two goods are otherwise identical. Therefore, her utility function is $U(X,Y) = X + 2Y$, where X is ounces of generic detergent and Y is ounces of name-brand detergent. The price of generic detergent is P_X and the price of name-brand detergent is P_Y. Income is I. Derive Maureen's demand curves for X and for Y as a function of P_X, P_Y, and I.

4. Suppose that there are two goods, X and Y. The utility function is $U(X,Y) = 10XY^2$. The price of Y is \$1 per unit, and the price of X is P. Derive the Hicksian demand curve for X for utility equal to 40,000 units.