Chapter 6 Online Appendix:
The Calculus of Production Functions and Input Demands

We saw in the chapter that production choices may be represented by production functions or by isoquants. In this appendix, we will be more explicit about how to derive isoquants directly from the production function.

Let's start with a specific production function to see how to approach deriving an isoquant. Suppose that a firm has the production function given by $Q(K,L) = K^{0.5}L^{0.5}$, where Q is the quantity of output, K is the amount of capital, and L is the amount of labor. An isoquant is the set of all the combinations of K and L that produce a certain amount of output. For our example, let's consider the isoquant for $Q = 10$ so that everywhere on the $Q = 10$ isoquant, $Q(K,L) = K^{0.5}L^{0.5} = 10$.

We could graph this function with K on the vertical axis and L on the horizontal axis, but it's often easier to graph the function when it is in the form where K is a function of L. To do this, we need to solve for K:

$$K^{0.5}L^{0.5} = 10$$

$$K^{0.5} = 10L^{-0.5}$$

Square both sides (recognizing that both K and L are nonnegative):

$$K = 100L^{-1} = \frac{100}{L}$$

Then graph this function to show the firm's $Q = 10$ isoquant. This is drawn in Figure 6OA.1.

Notice that the magnitude of the slope of this isoquant decreases as the firm uses more labor and less capital, holding output constant at 10 units. In other words, the marginal rate of technical substitution decreases as the firm moves along the isoquant. We can see the same result if we derive the $MRTS$ directly from the production function. Recall from the chapter that

$$MRTS_{LK} = \frac{MP_L}{MP_K}$$

Figure 6OA.1 A Firm's Isoquant when $Q = 10$
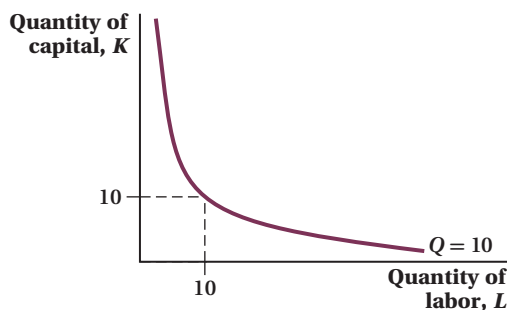
For our example,

$$MRTS_{LK} = \frac{MP_L}{MP_K} = \frac{\dfrac{\partial Q(K,L)}{\partial L}}{\dfrac{\partial Q(K,L)}{\partial K}} = \frac{0.5K^{0.5}L^{-0.5}}{0.5K^{-0.5}L^{0.5}} = \frac{K}{L} = KL^{-1}$$

To see how the marginal rate of substitution changes as the firm increases the quantity of labor, we can take the partial derivative of $MRTS_{LK}$ with respect to L:

$$\frac{\partial MRTS_{LK}}{\partial L} = \frac{\partial(KL^{-1})}{\partial L} = -KL^{-2} < 0$$

which is less than zero when K and L are positive. Therefore, $MRTS_{LK}$ decreases as the firm substitutes labor for capital along the isoquant. In other words, as the firm continues to substitute labor for capital, the same amount of capital needs to be replaced by ever larger amounts of labor in order to keep output constant.

Deriving the isoquant directly from the production function can also help us see why we get linear isoquants when inputs are perfectly substitutable. Suppose that a firm can produce a unit of output with either 1 unit of capital or 2 units of labor. The production function for this firm is $Q(K,L) = K + 0.5L$. To graph the $Q = 10$ isoquant for this production function, first solve for K as a function of L:

$$Q(K,L) = K + 0.5L = 10$$

$$K = 10 - 0.5L$$

This isoquant is now written in the familiar slope-intercept form of a linear equation in which the vertical intercept is 10 and the slope is −0.5. We can now graph this isoquant, as shown in Figure 6OA.2.

The vertical intercept tells us how many units of capital the firm needs to produce 10 units of output when no labor is used, and the horizontal intercept indicates how many units of labor the firm needs when no capital is used to produce 10 units of output. The negative of the slope is the $MRTS$ of labor for capital or the rate at which the firm is able to trade off capital and labor while keeping output unchanged. As above, we can also derive $MRTS_{LK}$ directly from the production function:

$$MRTS_{LK} = \frac{MP_L}{MP_K} = \frac{\dfrac{\partial Q(K,L)}{\partial L}}{\dfrac{\partial Q(K,L)}{\partial K}} = \frac{0.5}{1} = 0.5$$

The $MRTS_{LK}$ for this production function is constant because the marginal products of each input do not change as more of that input is used. In this case, capital and labor are perfect substitutes.

Figure 6OA.2 : **A Firm with a Linear Isoquant**
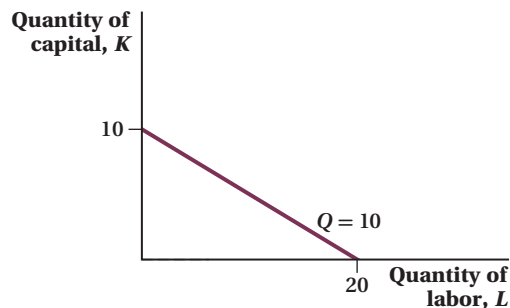
Demand for Inputs

Chapter 6 explores how changes in input prices affect the cost-minimizing combination of capital and labor using a graphical approach. When the wage increases and the isocost gets steeper (as in Figure 6.11), firms move up along the isoquant, using less labor and more capital. The graphical analysis is limited to particular levels of output and to discrete changes in input prices. Using calculus, we can extend the chapter's graphical analysis to derive the demand curves for inputs.

Consider a firm with our familiar Cobb–Douglas production function $Q(K,L) = AK^\alpha L^{1-\alpha}$, where $0 < \alpha < 1$ and $A > 0$. Because we are interested in the relationship between the prices of inputs and the quantities of inputs used by the firm, let's allow the wage (W), the rental rate of capital (R), and the level of output (Q) to vary. We can now set up the firm's cost-minimization problem as

$$\min_{K,L} RK + WL \text{ s.t. } Q = AK^\alpha L^{1-\alpha}$$

and write the problem as a Lagrangian:

$$\min_{K,L,\lambda} \mathcal{L}(K,L,\lambda) = RK + WL + \lambda(Q - AK^\alpha L^{1-\alpha})$$

The first-order conditions are

$$\frac{\partial \mathcal{L}}{\partial K} = R - \lambda(\alpha AK^{\alpha-1}L^{1-\alpha}) = 0$$

$$\frac{\partial \mathcal{L}}{\partial L} = W - \lambda[(1-\alpha)AK^\alpha L^{-\alpha}] = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = Q - AK^\alpha L^{1-\alpha} = 0$$

Rearrange the first two conditions to solve for λ:

$$R = \lambda(\alpha AK^{\alpha-1}L^{1-\alpha})$$

$$\lambda = \frac{R}{\alpha AK^{\alpha-1}L^{1-\alpha}}$$

$$W = \lambda[(1-\alpha)AK^\alpha L^{-\alpha}]$$

$$\lambda = \frac{W}{(1-\alpha)AK^\alpha L^{-\alpha}}$$

Now set these two expressions for λ equal to one another:

$$\lambda = \frac{W}{(1-\alpha)AK^\alpha L^{-\alpha}} = \frac{R}{\alpha AK^{\alpha-1}L^{1-\alpha}}$$

and solve for K as a function of L:

$$\frac{\alpha WL^{1-\alpha}}{(1-\alpha)RL^{-\alpha}} = \frac{K^\alpha A}{K^{\alpha-1}A}$$

$$K = \left(\frac{\alpha}{1-\alpha}\frac{W}{R}\right)L$$

Plug K as a function of L into the third first-order condition (the constraint):

$$Q - AK^\alpha L^{1-\alpha} = Q - A\left[\left(\frac{\alpha}{1-\alpha}\frac{W}{R}\right)L\right]^\alpha L^{1-\alpha} = Q - A\left(\frac{\alpha}{1-\alpha}\frac{W}{R}\right)^\alpha L^\alpha L^{1-\alpha} = 0$$

Now solve for the cost-minimizing quantity of labor as a function of the wage, holding output and the rental rate of capital constant:

$$A\left(\frac{\alpha}{1-\alpha}\frac{W}{R}\right)^\alpha L^\alpha L^{1-\alpha} = A\left(\frac{\alpha}{1-\alpha}\frac{W}{R}\right)^\alpha L = Q$$

$$L = L(W;R, Q) = \frac{Q}{A}\left(\frac{\alpha}{1-\alpha}\frac{W}{R}\right)^{-\alpha} = \frac{Q}{A}\left(\frac{1-\alpha}{\alpha}\frac{R}{W}\right)^\alpha$$

The demand for labor may also be written as

$$L = L(W; R, Q) = W^{-\alpha}\frac{Q}{A}\left(\frac{1-\alpha}{\alpha}R\right)^{\alpha}$$

We can show that the demand for labor satisfies the Law of Demand by taking the partial derivative with respect to the wage:

$$\frac{\partial L(W;R,Q)}{\partial W} = -W^{-\alpha-1}\frac{Q}{A}\left(\frac{1-\alpha}{\alpha}R\right)^{\alpha}$$

Because this is less than zero, the Law of Demand therefore holds.

To find the demand for capital, substitute the demand for labor into our expression for K as a function of L from above:

$$K(R;W,Q) = \left(\frac{\alpha}{1-\alpha}\frac{W}{R}\right)L = \left(\frac{\alpha}{1-\alpha}\frac{W}{R}\right)\frac{Q}{A}\left(\frac{1-\alpha}{\alpha}\frac{R}{W}\right)^{\alpha}$$

To simplify, invert the second term and combine:

$$K = K(R;W,Q) = \frac{Q}{A}\left(\frac{\alpha}{1-\alpha}\frac{W}{R}\right)\left(\frac{\alpha}{1-\alpha}\frac{W}{R}\right)^{-\alpha} = \frac{Q}{A}\left(\frac{\alpha}{1-\alpha}\frac{W}{R}\right)^{1-\alpha}$$

The demand for capital may also be written as

$$K(R;W,Q) = R^{-(1-\alpha)}\frac{Q}{A}\left(\frac{\alpha}{1-\alpha}W\right)^{1-\alpha}$$

Take the partial derivative of the demand for capital with respect to R:

$$\frac{\partial K(R;W,Q)}{\partial R} = -(1-\alpha)R^{-(2-\alpha)}\frac{Q}{A}\left(\frac{\alpha}{1-\alpha}W\right)^{1-\alpha} < 0$$

to show that the rental return for capital and the quantity demanded are inversely related and that the Law of Demand applies to the demand for capital.

6OA.1 figure it out

A firm has the production function $Q = 20K^{0.2}L^{0.8}$, where Q measures output, K represents machine hours, and L measures labor hours. Suppose that the wage rate is \$10 and the firm wants to produce 40,000 units of output.

a. Derive the demand for capital.

b. Confirm that the demand for capital satisfies the Law of Demand.

c. What is the optimal level of capital if the rental rate on capital is \$15?

Solution:

a. We could solve this problem using a Lagrangian; however, let's apply the cost-minimization condition instead. We know that the firm will minimize costs when

$$MRTS_{LK} = \frac{MP_L}{MP_K} = \frac{W}{R}$$

For this problem, the marginal rate of technical substitution is

$$MRTS_{LK} = \frac{MP_L}{MP_K} = \frac{(0.8)20K^{0.2}L^{-0.2}}{(0.2)20K^{-0.8}L^{0.8}} = \frac{4K}{L}$$

Applying the cost-minimization condition gives us

$$MRTS_{LK} = \frac{4K}{L} = \frac{W}{R} = \frac{10}{R}$$

Now solve for L as a function of K:

$$\frac{4K}{L} = \frac{10}{R}$$

$$4KR = 10L$$

$$L = 0.4KR$$

Substitute into the output constraint, and solve for K as a function of R:

$$40{,}000 = 20K^{0.2}L^{0.8}$$

$$40{,}000 = 20K^{0.2}(0.4KR)^{0.8}$$

$$40{,}000 = 20(0.4)^{0.8}R^{0.8}K^{0.2}K^{0.8}$$

$$40{,}000 \approx 20(0.48)R^{0.8}K$$

$$K \approx 4162.8R^{-0.8}$$

So, the demand for capital is

$$K(R) \approx 4162.8R^{-0.8}$$

b. Take the derivative of capital demand with respect to the rental rate:

$$\frac{\partial K(R)}{\partial R} = (-0.8)4162.8R^{-1.8} = -3330.24R^{-1.8}$$

which is negative, as we expect from the Law of Demand.

c. Substitute \$15 for R in the capital demand:

$$K \approx 4162.8R^{-0.8} \approx 4162.8(15)^{-0.8} \approx 477 \text{ machine hours}$$

This is exactly the result we found in Figure It Out 6A.1 in the text appendix to Chapter 6.

Problems

1. For the following production functions,
   - Write an equation and graph the isoquant for $Q = 100$.
   - Find the marginal rate of technical substitution and discuss how $MRTS_{LK}$ changes as the firm uses more L, holding output constant.
   a. $Q(K,L) = 2K + 3L$
   b. $Q(K,L) = K^{0.5}L^{0.25}$
   c. $Q(K,L) = LK + L$

2. Speedy Printing uses photocopiers (K) and printers (L). Its production function is $Q = K^{0.75}L^{0.25}$, where Q is the number of pages per day. Suppose that the rental rate on copiers is \$48 per day and that Speedy wants to produce 1,000 pages per day.
   a. Derive the demand for labor.
   b. Show that the demand for labor satisfies the Law of Demand.

3. A firm has the production function $Q = K^{0.5}L^{0.5}$. The wage is W and the rental rate on capital is R. Derive the demands for capital and labor as a function of Q, W, and R.