Chapter 7 Online Appendix: The Calculus of a Firm's Cost Structure Expanded

Marginal cost, average total cost, and average variable cost are all derived from total cost; therefore, these measures of a firm's costs are all related. Specifically, average cost is increasing when marginal cost is greater than average cost, average cost is decreasing when marginal cost is less than average cost, and average cost equals marginal cost when average cost is minimized. This chapter explains the relationship between marginal cost, average cost, and average variable cost by appealing to your intuition, but we can also use our knowledge of calculus to derive these relationships directly from the total cost function.

Suppose that total cost as a function of Q is given by $TC(Q)$ and total variable cost is $VC(Q)$. Then we know that marginal cost is

$$MC = \frac{dTC}{dQ} = \frac{dVC}{dQ}$$

and average total cost is

$$ATC = \frac{TC}{Q}$$

and average variable cost is

$$AVC = \frac{VC}{Q}$$

To find the quantity that minimizes average total cost, set the first derivative of ATC with respect to Q equal to zero:

$$\frac{dATC}{dQ} = \frac{d\left(\frac{TC}{Q}\right)}{dQ} = \frac{d(Q^{-1}TC)}{dQ} = 0$$

Apply the product rule from calculus:

$$\frac{dATC}{dQ} = \frac{d(Q^{-1})}{dQ}TC + \frac{dTC}{dQ}Q^{-1} = -Q^{-2}TC + \frac{dTC}{dQ}Q^{-1} = 0$$

or

$$Q^{-2}TC = \frac{dTC}{dQ}Q^{-1}$$

Let's rewrite the term on the left-hand side to express it in terms of average total cost:

$$Q^{-2}TC = Q^{-1}\frac{TC}{Q} = Q^{-1}ATC = \frac{ATC}{Q}$$

Rewrite the right-hand side to show marginal cost:

$$\frac{dTC}{dQ}Q^{-1} = Q^{-1}MC = \frac{MC}{Q}$$

Now we can rewrite the first order condition for minimizing average total cost as a function of ATC and MC:

$$\frac{dATC}{dQ} = -\frac{ATC}{Q} + \frac{MC}{Q} = \frac{1}{Q}(MC - ATC) = 0$$

We know that $\frac{1}{Q}$ cannot equal zero. Therefore, ATC is minimized at the quantity where

$$MC = ATC$$

We can see that ATC is minimized at this quantity by checking the second derivative. Recall that a function is at a minimum when the second derivative is positive. Let's take the second derivative of ATC with respect to Q:

$$\frac{d^2ATC}{dQ^2} = \frac{d\left(\frac{1}{Q}(MC - ATC)\right)}{dQ} = \frac{d\left(\frac{MC}{Q} - \frac{ATC}{Q}\right)}{dQ}$$
$$= -\frac{MC}{Q^2} + \frac{dMC}{dQ}\frac{1}{Q} - \left(-\frac{ATC}{Q^2} + \frac{dATC}{dQ}\frac{1}{Q}\right)$$

At the minimum of ATC, we already know that $MC = ATC$ and that $\frac{dATC}{dQ} = 0$. Therefore,

$$\frac{d^2ATC}{dQ^2} = -\frac{MC}{Q^2} + \frac{dMC}{dQ}\frac{1}{Q} + \frac{MC}{Q^2} - (0)\frac{1}{Q} = \frac{dMC}{dQ}\frac{1}{Q}$$

We know that $Q > 0$ so this second derivative is positive when

$$\frac{dMC}{dQ} > 0$$

Therefore, the average total cost and average variable cost curves are at their minimums only when the marginal cost curve is increasing at that quantity. Why does the slope of the marginal cost curve matter? If marginal cost is decreasing when it crosses the average cost curve, then average cost has to be decreasing at that quantity. If the next unit costs less to produce, then the average cost of producing more also has to be less, and average cost can't be at its lowest level.

We can also find the minimum of average variable cost to show that this happens where $AVC = MC$. Set the first derivative of AVC with respect to Q equal to zero:

$$\frac{dAVC}{dQ} = \frac{d\left(\frac{VC}{Q}\right)}{dQ} = \frac{d(Q^{-1}VC)}{dQ} = \frac{d(Q^{-1})}{dQ}VC + \frac{dVC}{dQ}Q^{-1} = -Q^{-2}VC + \frac{dVC}{dQ}Q^{-1} = 0$$

Then substitute AVC and MC where appropriate:

$$\frac{dAVC}{dQ} = -\frac{AVC}{Q} + \frac{MC}{Q} = \frac{1}{Q}(MC - AVC) = 0$$

Thus, average variable cost is minimized at the quantity where:

$$MC = AVC$$

As in our examination of average total cost above, the second-order condition requires MC to be increasing when it crosses the AVC curve to guarantee that this is the minimum of average variable cost.

The chapter discusses the intuition that average total cost or average variable cost is increasing when marginal cost is above the respective average cost curve and decreasing when marginal cost is less than the respective average cost. Let's examine the relationship between average total cost and marginal cost. Whether ATC is increasing or decreasing as Q increases is indicated by the sign of the first derivative of ATC with respect to Q. From above:

$$\frac{dATC}{dQ} = \frac{1}{Q}(MC - ATC)$$

which is positive (and ATC is increasing) if $MC > ATC$. This derivative is negative (and ATC is decreasing) if $MC < ATC$.

7OA.1 figure it out

Suppose a firm's total cost curve is $TC = 15Q^2 + 8Q + 45$.

a. Find the firm's marginal cost, average total cost, and average variable cost.

b. Use calculus to find the output level that minimizes average total cost.

c. Show that $ATC = MC$ at this output level.

d. Show that this is the minimum ATC by examining the second derivative.

Solution:

a. Marginal cost is

$$MC = \frac{dTC}{dQ} = \frac{d(15Q^2 + 8Q + 45)}{dQ} = 30Q + 8$$

Average total cost is

$$ATC = \frac{TC}{Q} = \frac{15Q^2 + 8Q + 45}{Q} = 15Q + 8 + \frac{45}{Q}$$

To find average variable cost, we need to recognize that fixed cost is 45 and $VC = 15Q^2 + 8Q$. Then average variable cost is

$$AVC = \frac{VC}{Q} = \frac{15Q^2 + 8Q}{Q} = 15Q + 8$$

b. To find the minimum of ATC, set its first derivative equal to zero and solve for Q:

$$\frac{dATC}{dQ} = \frac{d\left(15Q + 8 + \frac{45}{Q}\right)}{dQ} = 15 - \frac{45}{Q^2} = 0$$

$$15 = \frac{45}{Q^2}$$

$$15Q^2 = 45$$

$$Q^2 = 3$$

$$Q = \sqrt{3} \approx 1.732$$

Alternatively, we could find the quantity where $MC = ATC$:

$$MC = 30Q + 8 = 15Q + 8 + \frac{45}{Q} = ATC$$

$$15Q = \frac{45}{Q}$$

$$15Q^2 = 45$$

$$Q^2 = 3$$

$$Q = \sqrt{3} \approx 1.732$$

c. At this quantity, average total cost is

$$ATC \approx 15(1.732) + 8 + \frac{45}{1.732} \approx 59.96$$

and marginal cost is

$$MC \approx 30(1.732) + 8 \approx 59.96$$

d. Take the second derivative of average total cost:

$$\frac{d^2ATC}{dQ^2} = \frac{d\left(15 - \dfrac{45}{Q^2}\right)}{dQ} = \frac{90}{Q^3} > 0$$

for any $Q > 0$ so this is the minimum of ATC.

Problems

1. Candy's Tortilla Company has a total cost function given by $TC = Q^2 + 4Q + 9$.
   a. What are Candy's average total cost, average variable cost, and marginal cost?
   b. Find the quantity that minimizes ATC.
   c. Show that $ATC = MC$ when ATC is minimized.

2. Suppose that Bob's Donuts has a total cost function given by $TC = Q^3 - 6Q^2 + 14Q + 5$. Find the quantity that minimizes average variable costs for Bob's Donuts. Demonstrate that average variable cost is equal to marginal cost at this quantity.