Chapter 11 Online Appendix:
The Calculus of Cournot and Differentiated Bertrand Competition Equilibria

In this appendix, we explore the Cournot and Bertrand market structures. The textbook describes the key divergence between these two market structures that at first glance seem very similar. The subtle difference is that firms in a Cournot market environment do not compete by setting price as they do in the Bertrand market setting. Rather, firms in a Cournot market structure interact by setting output. Despite the close relationship between price and quantity in markets, the outcome in a market structure in which firms compete in price (Bertrand) is often dramatically different from the outcome when firms compete in output choice (Cournot).

We begin by investigating a Cournot market setting and demonstrate the loss in efficiency common to this market structure. The textbook chapter presents the Cournot analysis in the special case of linear demand curves and linear cost functions. In this appendix, we allow for cubic cost functions in the Cournot setting and also identify the efficiency impact of more firms entering a Cournot industry with cubic cost functions.

Cournot Market Structure

Suppose that there are two producers of organic farm products in the local market. The producers are Applewood Farm and River Bottom Farm. Each of these farms produces homogeneous organic sweet peas that are purchased by local buyers. The inverse demand function for organic sweet peas is

$$P = 10 - 0.005Q$$

where P is the price per pound of sweet peas and Q is the local market output of sweet peas in pounds. We will denote the production of sweet peas by Applewood Farm in pounds as q_A and the production of sweet peas by River Bottom Farm in pounds as q_R. Because these two producers are the only producers in the local market, market output is

$$Q = q_A + q_R$$

The cost for sweet pea production at Applewood Farm is given by

$$C_A(q_A) = 0.00024q_A^3 - 0.048q_A^2 + 3.4q_A$$

The average variable cost curve, which is identified by dividing $C_A(q_A)$ by q_A, is

$$AVC_A(q_A) = 0.00024q_A^2 - 0.048q_A + 3.4$$

The average variable cost curve attains its minimum where the slope of the curve equals zero. The slope of the average variable cost curve is found by taking the derivative of average variable cost with respect to q_A. That is,

$$0 = \frac{dAVC_A}{dq_A} = 0.00048q_A - 0.048$$

Solving for q_A, we observe that the average variable cost curve takes on its minimum value when $q_A = 100$. Average variable cost at its minimum is

$$AVC_A(q_A = 100) = 0.00024(10{,}000) - 0.048(100) + 3.4 = 1$$

The price for organic sweet peas must be at or above \$1 per pound to ensure that Applewood Farm will operate in the short run.

The cost for sweet pea production at River Bottom Farm is given by

$$C_R(q_R) = 0.00024q_R^3 - 0.048q_R^2 + 3.4q_R$$

As we confirmed for Applewood Farm, the equivalent cost structure for River Bottom Farm suggests that the average variable cost for River Bottom Farm also occurs at $q_R = 100$. Moreover, average variable cost at its minimum value is \$1.00. To induce River Bottom Farm to operate in the short run, its price must be at or above \$1 per pound of organic sweet peas.

Applewood Farm Behavior in a Cournot Market Structure From the information provided, we can write Applewood Farm's profit function as

$$\pi_A = \big(10 - 0.005(q_A + q_R)\big)q_A - (0.00024q_A^3 - 0.048q_A^2 + 3.4q_A)$$

Applewood Farm's objective is to maximize π_A given its choice of q_A. Applewood's first-order condition is

$$0 = \frac{\partial \pi_A}{\partial q_A} = 10 - 0.01q_A - 0.005q_R - 0.00072q_A^2 + 0.096q_A - 3.4$$

Simplifying, we may write the first-order condition for Applewood's profit maximization in the form of the quadratic equation. That is,

$$0 = -0.00072q_A^2 + 0.086q_A + 6.6 - 0.005q_R$$

Using the quadratic formula, we note that the two values of q_A which satisfy the quadratic equation are

$$q_A = 59.72 \pm \frac{\sqrt{0.026404 - 0.0000144q_R}}{-0.00144}$$

Taking the second partial derivative of Applewood's profit function with respect to q_A, Applewood Farm's second-order sufficient condition for maximization is

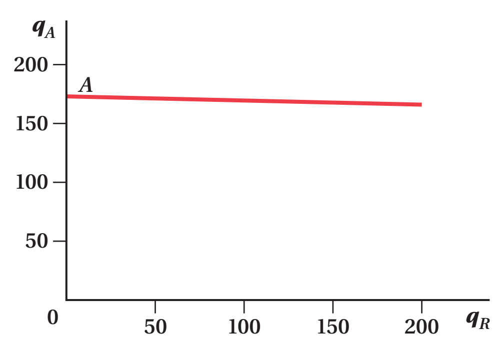
$$\frac{\partial^2 \pi_A}{\partial q_A^2} = 0.086 - 0.00144q_A < 0$$

This implies that the choice of output that maximizes profits corresponds to $q_A > 59.72$. Recognizing this and using the result from our quadratic equation, we reason that Applewood Farm's profit-maximizing reaction curve to River Bottom Farm's output choice is

$$q_A = 59.72 + \frac{\sqrt{0.026404 - 0.0000144q_R}}{0.00144}$$

Figure 11OA.1 shows Applewood Farm's reaction curve.

Figure 11OA.1 Applewood Farm Reaction Curve



River Bottom Farm Behavior in a Cournot Market Structure From the information provided, we can express River Bottom Farm's profit function as

$$\pi_R = \big(10 - 0.005(q_A + q_R)\big)q_R - (0.00024q_R^3 - 0.048q_R^2 + 3.4q_R)$$

River Bottom Farm's objective is to maximize π_R given its choice of q_R. Its first-order condition is

$$0 = \frac{\partial \pi_R}{\partial q_R} = 10 - 0.01q_R - 0.005q_A - 0.00072q_R^2 + 0.096q_R - 3.4$$

Simplifying, we may write the first-order condition for profit maximization in the form of the quadratic equation. That is,

$$0 = -0.00072q_R^2 + 0.086q_R + 6.6 - 0.005q_A$$

Using the quadratic formula, we note that the two values of q_R which satisfy the quadratic equation are

$$q_R = 59.72 \pm \frac{\sqrt{0.026404 - 0.0000144q_A}}{-0.00144}$$

Taking the second partial derivative of River Bottom Farm's profit function with respect to q_R, we identify its second-order sufficient condition for maximization as

$$\frac{\partial^2 \pi_R}{\partial q_R^2} = 0.086 - 0.00144q_R < 0$$

This implies that the choice of output that maximizes profits corresponds to $q_R > 59.72$. Recognizing this and using the result from our quadratic equation, we reason that River Bottom Farm's profit-maximizing reaction curve to Applewood Farm's output choice is

$$q_R = 59.72 + \frac{\sqrt{0.026404 - 0.0000144q_A}}{0.00144}$$

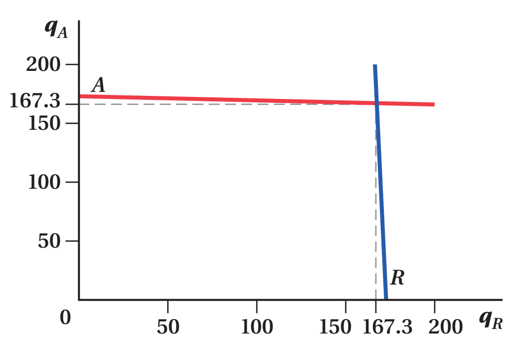
Figure 11OA.2 shows River Bottom Farm's reaction curve in the same diagram as Applewood Farm's reaction curve.

Identifying Equilibrium Choices in a Cournot Market Structure In our local market with two farms, we identified Applewood Farm's first-order condition after combining like terms as

$$0 = 6.6 - 0.0005q_R - 0.00072q_A^2 + 0.086q_A$$

In deriving Applewood Farm's reaction curve above, we treated q_R as if it were an unknown constant that will be realized when Applewood Farm makes its choice of q_A. A subtlety of the Cournot market structure is that the farms make their choices

Figure 11OA.2 Sweet Peas Market Reaction Curves



independently; that is, without knowing the choice of the other farm. However, if we suppose that both farms have full information (e.g., each farm knows the demand curve and cost structure of the other farm), each farm is able to anticipate the first-order condition of the other farm. Next, if we recognize that each farm's cost structure for producing sweet peas is equivalent to the other, the farms must also realize that $q = q_A = q_R$. Making this substitution in Applewood Farm's first-order condition, we obtain the quadratic equation

$$0 = -0.00072q^2 + 0.081q + 6.6$$

Applying the quadratic formula, we recognize that

$$q = \frac{0.081}{0.00144} \pm \frac{\sqrt{0.006561 + 0.019008}}{-0.00144}$$

The two possible values of q that satisfy the quadratic formula are $q = -54.8$ or $q = 167.3$. Given that output must be positive, we conclude that equilibrium in the local sweet peas market will result in each farm producing 167.3 pounds of organic sweet peas. Inspecting Figure 11OA.2 above, we see that this equilibrium corresponds to the combination of production by the farms such that both farms are on their reaction curve. This point satisfies our notion of equilibrium in a Cournot market: Neither farm has an incentive to change its behavior given the output choice of the other farm.

Total output in the local organic sweet pea market is 334.6 pounds. At this output level, the price in the market is $P = 10 - 0.005Q = 10 - 0.005(334.6) = 8.327$. That is, the price per pound of organic sweet peas is approximately \$8.33.

Price-Taking Behavior and the Market Outcome with Two Farms In this section, we compare the Cournot market outcome to an otherwise identical industry of two farms behaving as price takers. We demonstrate that total surplus is maximized (i.e., the market outcome is efficient) when the two farms behave as price takers. Total surplus is maximized at the output level at which marginal benefit is equal to marginal cost. Symbolically, this is $MB(Q) = MC(Q)$.

The marginal cost of sweet pea production for Applewood Farm is $MC_A(q_A) = \frac{dC_A}{dq_A} = 0.00072q_A^2 - 0.096q_A + 3.4$. If Applewood behaved as a price taker, it would choose the level of output at which $P = MC_A$. Thus, we reason that Applewood Farm behaving as a price taker would choose the level of output such that

$$P = 0.00072q_A^2 - 0.096q_A + 3.4$$

In the form of a quadratic equation, this is

$$0 = 0.00072q_A^2 - 0.096q_A + 3.4 - P$$

Using the quadratic formula, we identify

$$q_A = 66.67 \pm \frac{\sqrt{0.00288P - 0.00058}}{0.00144}$$

Recognizing that $q_A > 66.67$ and simplifying, we have

$$q_A = 66.67 + 37.27\sqrt{P - 0.20}$$

In the first section discussing the Cournot market structure, we demonstrated that price must be at or above \$1 per pound of organic sweet peas to induce Applewood Farm to produce. Notice that when $P = 1$, the supply curve for Applewood acting as a price taker confirms that it would produce 100 pounds $\left(q_A = 66.67 + 37.27\sqrt{1 - 0.20} = 66.67 + 37.27(0.894) = 100\right)$.

Following the same steps we used above to identify Applewood Farm's supply curve when it acts as a price taker, we find the following for River Bottom Farm: $q_R = 66.67 + 37.27\sqrt{P - 0.20}$.

Using Applewood Farm's and River Bottom Farm's supply curves, we can derive the competitive supply curve:

$$Q^S = q_A + q_R = 133.33 + 74.54\sqrt{P - 0.20}$$

Solving this competitive supply curve for price, we find the inverse supply curve:

$$P = 0.20 + \left(\frac{Q - 133.33}{74.54}\right)^2$$

$$= 0.20 + \frac{Q^2 - 266.67Q + 17{,}777.78}{5{,}556.212}$$

$$= 0.00018Q^2 - 0.048Q + 3.4$$

Notice that we can plot this industry supply curve in price and quantity space with the demand curve identified in the first section giving the Cournot market structure. This market supply and demand model is presented in Figure 11OA.3.

This inverse competitive supply curve is also the marginal cost curve for the industry. That is, for a particular choice of output, say, Q_0, $P = MC(Q_0)$. With this marginal cost curve, we can identify the socially efficient output level with the equation

$$MB(Q) = 10 - 0.005Q = 0.00018Q^2 - 0.048Q + 3.4 = MC(Q)$$

We can manipulate this equation into the form of a quadratic equation as

$$0 = 0.00018Q^2 - 0.043Q - 6.6$$

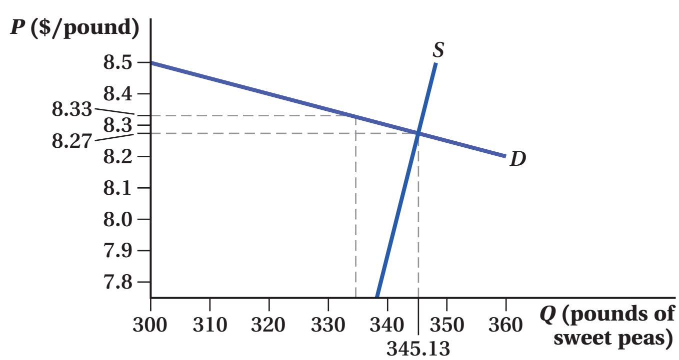
The quadratic formula identifies two possible solutions to the equation. They are

$$Q = 119.44 \pm 2{,}777.78\sqrt{0.006601}$$

$$= (-106.24, 345.13)$$

From our analysis of the average variable cost curves, we reason that the output level that achieves social efficiency with each firm behaving as a price taker is $Q = 345.13$. This industry output level implies a market price of $P = 10 - 0.005(345.13) = 8.27$. The equilibrium price given each firm's behavior as a price taker is \$8.27 per pound of sweet peas. This equilibrium is demonstrated by the intersection of the supply curve and market demand curve in Figure 11OA.3 below. The dashed gray horizontal line at $P = \$8.33$ presents the Cournot equilibrium price we discovered in the previous section. Firms in the Cournot market produce less output and charge a greater price relative to price-taking firms in a competitive market. From this comparison, we learn that there is a deadweight loss associated with the Cournot market structure with two farms.

Figure 11OA.3 Two Price-Taking Farms



The Number of Farms Operating and Price Taking Market Outcomes

Let's now consider how the efficient market outcome changes as we increase the number of farms operating in the industry. We assume that all farms operating in the industry have the same cost structure as Applewood Farm and River Bottom Farm. We write the number of farms operating in the industry as n. We can then write the output choice of any of the farms operating in the industry as q_i, where i is an integer from 1 to n. The supply curve for farm i is

$$q_i = 66.67 + 37.27\sqrt{P - 0.20}$$

The industry supply curve is then

$$Q^S = \sum_{i=1}^{n}(66.67 + 37.27\sqrt{P - 0.20}) = 66.67n + 37.27n\sqrt{P - 0.20}$$

By solving for P, we identify the inverse supply curve as

$$\begin{aligned} P &= 0.20 + \left(0.0268\frac{Q}{n} - 1.79\right)^2 \\ &= 0.20 + \left(0.00072\left(\frac{Q}{n}\right)^2 - 0.096\frac{Q}{n} + 3.20\right) \\ &= 0.00072\left(\frac{Q}{n}\right)^2 - 0.096\frac{Q}{n} + 3.40 \end{aligned}$$

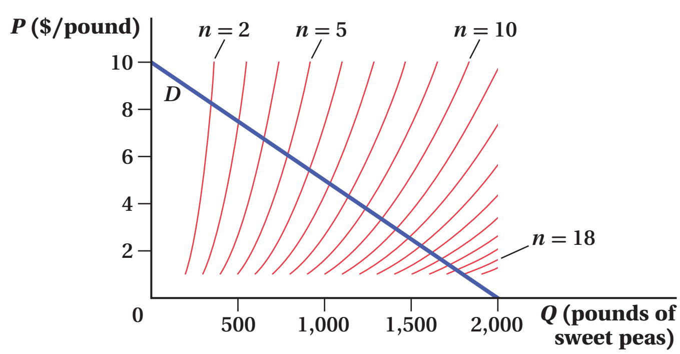
Figure 11OA.4 plots the industry supply curve for industry sizes from 2 to 19 farms with the industry demand curve. Each red industry supply curve corresponds to a different number of farms operating in the local organic sweet pea industry. The red curve furthest to the left and marked $n = 2$ corresponds to the two-farm Cournot industry. This industry supply curve is identical to the supply curve plotted in Figure 11OA.3 on the previous page.

Take note that total surplus is maximized when $n = 18$ and all 18 farms are behaving as price takers. In this competitive industry, each farm produces at the minimum average variable cost of $1 and produces 100 pounds of sweet peas. This results in industry output of 1,800 pounds of organic sweet peas and a market price of $1.00 per pound. Each farm earns zero producer surplus.

The Cournot Outcome with 18 Farms Now, let's consider the implication of allowing the 18 farms to operate in a Cournot market structure (and thus they no longer behave as price takers). We continue to assume that all farms that operate in the local market have the same cost structure. We can write farm i's profit-maximization problem as

$$\pi_i = \left(10 - 0.005(q_i + Q_i^o)\right)q_i - (0.00024q_i^3 - 0.048q_i^2 + 3.4q_i)$$

Figure 11OA.4 Price-Taking Farms and Market Outcomes (n = 2 to n = 19)



where Q_i^O is the combined output of all other farms in the market. The corresponding first-order condition is

$$0 = \frac{d\pi_i}{dq_i} = 10 - 0.01q_i - 0.005Q_i^o - 0.00072q_i^2 + 0.096q_i - 3.4$$

Rearranging the equation, we have

$$0 = -0.00072q_i^2 + 0.086q_i + 6.6 - 0.005Q_i^o$$

As we reasoned earlier, the equivalent cost structure across the farms implies that each farm will produce the same level of output. This suggests $Q_i^o = (n - 1)q$, where q is the level of production by each farm. Substituting for Q_i^o and q_i in the quadratic equation, we have

$$0 = -0.00072q^2 + 0.086q + 6.6 - 0.005(n - 1)q$$

Simplifying, we may write our quadratic equation as

$$0 = -0.00072q^2 + (0.091 - 0.005n)q + 6.6$$

Using the quadratic formula, possible solutions to the equation are given by

$$q = 63.19 - 3.47n \pm 694.44\sqrt{0.0273 - 0.00091n + 0.000025n^2}$$

For the case of $n = 18$, the solutions to the equation are

$$q = (-95.04, 96.50)$$

Given that output must be positive, we see each farm will choose to produce 96.50 pounds of sweet peas. Total industry output is 1,737 pounds. The market price is $P = 10 - 0.005(1{,}737) = 1.31$. Notice that 18 price-taking firms result in 1,800 pounds and a market price of \$1. It will take more than 18 firms in the Cournot market structure before 1,800 pounds are brought to the market.

Bertrand Market Structure with Differentiated Services and Nonlinear Costs

Let's turn now to the Bertrand market structure. We will look at this through a worked-out problem in a Figure It Out.

11OA.1 figure it out

This Figure It Out is similar to Figure It Out 11.4 in the textbook, except that here we use a quadratic cost function.

Cowboy Pete's Ranch and Wily Western Ranch are two dude ranches in the Midwest. The ranches offer horseback riding, sightseeing along winding rivers, a campfire experience, and cabin lodgings. The demand curve for a night at Cowboy Pete's ranch is given by

$$q_C = 400 - 7.5p_C + 5p_W$$

where p_C is the price charged for a night at Cowboy Pete's ranch and p_W is the price charged for a night's stay at the Wily Western Ranch. The demand curve for a night's stay at Wily Western Ranch is

$$q_W = 600 + 2.5p_C - 5p_W$$

The cost of providing a night at Cowboy Pete's is

$$C_C(q_C) = q_C^2$$

Wily Western Ranch provides a more authentic (and therefore more costly to maintain?) Western experience for customers. The cost of providing a night at Wily Western Ranch is

$$C_W(q_W) = 2q_W^2$$

a. Derive a price reaction function for each ranch.

b. Diagram the reaction function for each ranch in price space and indicate the equilibrium on the graph.

c. Mathematically, identify the equilibrium price charged by each dude ranch. Also, identify the equilibrium quantity of guest nights provided and profits by each dude ranch.

d. In this differentiated-demand Bertrand market structure, are the dude ranches setting the price equal to marginal cost? Demonstrate.

Solution:

a. Derive a price reaction function for each ranch. Let's begin with Cowboy Pete's Ranch. Cowboy Pete's profit can be written as

$$\pi_C(p_C, p_W) = 400p_C - 7.5p_C^2 + 5p_C p_W - C_C\big(q_C(p_C, p_W)\big)$$

In this expression for profit, total revenue is expressed as a function of prices at the dude ranches. We can also represent cost solely as a function of prices at the dude ranches: $C_C\big(q_C(p_C, p_W)\big) = q_C^2 = (400 - 7.5p_C + 5p_W)^2$.

Therefore, Cowboy Pete's profit is

$$\pi_C(p_C, p_W) = 400p_C - 7.5p_C^2 + 5p_C p_W - (400 - 7.5p_C + 5p_W)^2$$

Cowboy Pete's objective is to choose p_C to maximize profits. The corresponding first-order condition is

$$0 = \frac{d\pi}{dp_C} = 400 - 15p_C + 5p_W - 2(400 - 7.5p_C + 5p_W)(-7.5)$$

Simplifying and solving for p_C, we find

$$0 = 6{,}400 - 127.5p_C + 80p_W$$

$$p_C = 50.20 + 0.63p_W$$

This last function is Cowboy Pete's optimal reaction function to the price set by Wily Western Ranch. Now, let's identify Wily Western Ranch's reaction function to Cowboy Pete's price. Wily Western Ranch profit can be written as

$$\pi_W = 600p_W + 2.5p_C p_W - 5p_W^2 - 2(600 + 2.5p_C - 5p_W)^2$$

Notice that we have already substituted the demand function in the cost function, as we did for Cowboy Pete's Ranch. Wily Western Ranch's objective is to maximize profit with the ranch's choice of p_W. The resulting first-order condition is

$$0 = \frac{d\pi}{dp_W} = 600 + 2.5p_C - 10p_W - 4(600 + 2.5p_C - 5p_W)(-5)$$

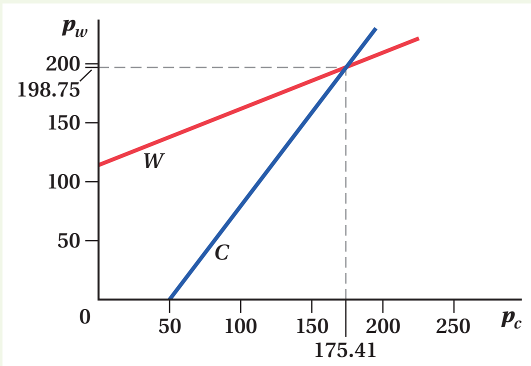
Simplifying and solving for p_W, we have

$$0 = 12{,}600 + 52.5p_C - 110p_W$$

$$p_W = 114.55 + 0.48p_C$$

This final function is Wily Western Ranch's reaction function to Cowboy Pete's price.

b. The figure below shows the reaction functions derived in part (a). Cowboy Pete's reaction function is drawn in blue; Wily Western Ranch's reaction function is drawn in red. Graphically, the equilibrium in this Bertrand market with differentiated dude ranch experiences occurs at the intersection of the reaction functions. Equilibrium occurs where Cowboy Pete's Ranch charges \$175.41 per night and Wily Western Ranch charges \$198.75 per night.



c. We can determine mathematically the equilibrium prices charged at each ranch by first noting the reaction functions

$$p_C = 50.20 + 0.63p_W$$

and

$$p_W = 114.55 + 0.48p_C$$

If we substitute Wily Western Ranch's reaction function in Cowboy Pete's reaction function, we have

$$p_C = 50.20 + 0.63(114.55 + 0.48p_C)$$

Simplifying and solving for p_C, we find

$$p_C = 175.41$$

Returning to Wily Western Ranch's reaction function and substituting $p_C = \$175.41$, we identify Wily Western Ranch's equilibrium price as \$198.75 per night.

The equilibrium quantity of nights for each ranch can be identified by inserting our equilibrium prices into each ranch's demand curve. For Cowboy Pete's, this is

$$q_C = 400 - 7.5(175.41) + 5(198.75) = 78.175$$

For Wily Western Ranch, we have

$$q_W = 600 + 2.5(175.41) - 5(198.75) = 44.775$$

Rounded to the nearest integer, we anticipate that Cowboy Pete's will have 78 guest nights each period, while Wily Western Ranch has 45 guest nights each period. Cowboy Pete's will generate revenue of \$13,681.98 and costs of \$6,084. Thus, Cowboy Pete's period profit is \$7,597.98. Wily Western Ranch generates revenues of \$8,943.75 and costs of \$4,050. Wiley Western Ranch's period profit is \$4,893.75.

d. Cowboy Pete's marginal cost function is

$$MC_C = \frac{dC_C}{dq_C} = 2q_C$$

Inserting Cowboy Pete's equilibrium production quantity determined in part (c), we have

$$p_C = 175.41 > 156 = 2(78) = MC_C$$

Cowboy Pete's ranch sets its price above marginal cost.

Wily Western Ranch's marginal cost function is

$$MC_W = \frac{dC_W}{dq_W} = 4q_W$$

Inserting Wily Western Ranch's equilibrium production from part (c), we find

$$p_W = 198.75 > 180 = 4(45) = MC_W$$

That is, Wily Western Ranch also sets its price above marginal cost. In this Bertrand market structure with differentiated services, we verify that each firm has pricing power.

Problems

1. The inverse demand for sandwiches over the lunch hour is given by $P = 8 - 0.01Q$, where P is the price per sandwich and Q is the total number of sandwiches brought to market. There are two sandwich shops operating in the local market. Firm 1's cost function is $C_1 = 0.0002q_1^2$, where q_1 is the number of sandwiches brought to market by Firm 1. Firm 2's cost function is $C_2 = 0.0002q_2^2$, where q_2 is the number of sandwiches brought to market by Firm 2. Given that the two firms compete by setting output (Cournot), answer the following:
   a. Write Firm 1's profit function. Note that $Q = q_1 + q_2$.
   b. Identify Firm 1's reaction function to Firm 2's output.
   c. Identify Firm 2's reaction function to Firm 1's output.
   d. Identify the equilibrium output level of each firm and the equilibrium price for sandwiches.

2. Ray's Diner offers traditional deli foods to lunch customers in the local market. On its lunch menu, Flo's Eatery offers more variety. The two establishments compete in a Bertrand market structure with differentiated products. Ray's Diner is confronted with the demand curve for its daily special given by $q_R = 100 - 10p_R + 5p_F$, where p_R is the price for the daily special at Ray's Diner and p_F is the price for the daily special at Flo's Eatery. Flo's faces the demand curve for the daily special given by $q_F = 200 - 10p_F + 2p_R$. The cost of providing the special at Ray's Diner is $C_R = q_R$, while the cost of providing the special at Flo's Eatery is $C_F = 3q_F$.
   a. Identify for Ray's Diner the profit function from providing the special.
   b. Identify for Ray's Diner the reaction function to the Eatery's price for the special.
   c. Identify for Flo's Eatery the reaction function to the Diner's price for the special.
   d. Identify the equilibrium price charged for a special at Ray's Diner and Flo's Eatery.