∂ 6.4 figure it out

For each of the following production functions, determine if they exhibit constant, decreasing, or increasing returns to scale.

a. $Q = 2K + 15L$

b. $Q = \min(3K, 4L)$

c. $Q = 15K^{0.5}L^{0.4}$

d. For the firm in part (c), find the marginal rate of technical substitution and discuss how $MRTS_{LK}$ changes as the firm uses more L, holding output constant.

e. Suppose that the wage rate is \$10 per hour and the rental rate of capital is \$25 per hour. If the firm in part (c) wants to produce 10,000 units of output, what is the cost-minimizing bundle of capital and labor?

f. Derive the firm in part (c)'s expansion path.

g. Derive the firm in part (c)'s demand for labor.

h. Confirm that the demand for labor for the firm in part (c) satisfies the Law of Demand.

Solution:

The easiest way to determine the returns to scale for a production function is to simply plug in values for L and K, calculate Q, and then double the input levels to see what happens to output. If output exactly doubles, the production function exhibits constant returns to scale. If output rises by less than double, there are decreasing returns to scale. If output more than doubles, the production function has increasing returns to scale.

So, for each of these production functions, we will start with $K = L = 1$ and calculate Q and then perform the same exercise for $K = L = 2$. Note that K and L do not have to be equal for this method to work, but it does simplify the solution a bit.

a. If $L = 1$ and $K = 1$: $Q = 2K + 15L = 2(1) + 15(1) = 2 + 15 = 17$.
If $L = 2$ and $K = 2$: $Q = 2K + 15L = 2(2) + 15(2) = 4 + 30 = 34$.

Since output exactly doubles when inputs are doubled, the production function exhibits constant returns to scale.

b. If $L = 1$ and $K = 1$: $Q = \min(3K, 4L) = Q = \min(3(1), 4(1)) = \min\ (3, 4) = 3$.
If $L = 2$ and $K = 2$: $Q = \min(3K, 4L) = Q = \min(3(2), 4(2)) = \min\ (6, 8) = 6$.

Because output exactly doubles when inputs are doubled, the production function exhibits constant returns to scale.

c. If $L = 1$ and $K = 1$: $Q = 15K^{0.5}L^{0.4} = Q = 15(1)^{0.5}(1)^{0.4} = 15(1)(1) = 15$.

If $L = 2$ and $K = 2$: $Q = 15K^{0.5}L^{0.4} = Q = 15(2)^{0.5}(2)^{0.4} = 15(1.41)(1.31) = 27.71$.

Because output less than doubles when inputs are doubled, the production function exhibits decreasing returns to scale.

d. Use the marginal products to solve for the $MRTS$:

$$MRTS_{LK} = \frac{MP_L}{MP_K} = \frac{\frac{\partial Q}{\partial L}}{\frac{\partial Q}{\partial K}} = \frac{0.4(15)K^{0.5}L^{-0.6}}{0.5(15)K^{-0.5}L^{0.4}} = \frac{4K}{5L} = 0.8KL^{-1}$$

We can take the partial of the $MRTS_{LK}$ with respect to L to see how the $MRTS_{LK}$ changes as L increases:

$$\frac{\partial MRTS_{LK}}{\partial L} = -0.8KL^{-2} < 0$$

Thus, the $MRTS_{LK}$ declines as the firm uses more labor.

e. First, we set up the firm's cost-minimization problem as

$$\min_{K,L} 25K + 10L \text{ s.t. } 10{,}000 = 15K^{0.5}L^{0.4}$$

or

$$\min_{K,L,\lambda} \mathcal{L}(K,L,\lambda) = 25K + 10L + \lambda(10{,}000 - 15K^{0.5}L^{0.4})$$

Find the first-order conditions for the Lagrangian:

$$\frac{\partial \mathcal{L}}{\partial K} = 25 - \lambda\left(15(0.5)K^{-0.5}L^{0.4}\right) = 0$$

$$\frac{\partial \mathcal{L}}{\partial L} = 10 - \lambda\left(15(0.4)K^{0.5}L^{-0.6}\right) = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = 10{,}000 - 15K^{0.5}L^{0.4} = 0$$

Solve for L as a function of K using the first two conditions:

$$\lambda = \frac{25}{15(0.5)K^{-0.5}L^{0.4}} = \frac{10}{15(0.4)K^{0.5}L^{-0.6}}$$

$$150K^{0.5}L^{-0.6} = 75K^{-0.5}L^{0.4}$$

$$2(K^{0.5}K^{0.5}) = (L^{0.6}L^{0.4})$$

$$L = 2K$$

Now plug L into the third first-order condition and solve for the optimal number of labor and machine hours, L^* and K^*:

$$10{,}000 = 15K^{0.5}L^{0.4}$$

$$10{,}000 = 15K^{0.5}(2K)^{0.4}$$

$$15(2)^{0.4}K^{0.9} = 10{,}000$$

$$K^{0.9} \approx 505.24$$

$$K^* \approx 1{,}009 \text{ machine hours}$$

$$L^* = 2(1{,}009) = 2{,}018 \text{ labor hours}$$

f. The firm's expansion path reveals the relationship between the optimal amounts of labor and capital input. Within the Lagrangian procedure in part (e), we found that $L = 2K$ or $K = 0.5L$. This is the firm's expansion path.

g. We set up the firm's cost-minimization problem as before but this time with a generic wage rate W:

$$\min_{K,L} 25K + WL \text{ s.t. } 10{,}000 = 15K^{0.5}L^{0.4}$$

or

$$\min_{K,L,\lambda} \mathcal{L}(K,L,\lambda) = 25K + WL + \lambda(10{,}000 - 15K^{0.5}L^{0.4})$$

Find the first-order conditions for the Lagrangian:

$$\frac{\partial \mathcal{L}}{\partial K} = 25 - \lambda\left(15(0.5)K^{-0.5}L^{0.4}\right) = 0$$

$$\frac{\partial \mathcal{L}}{\partial L} = W - \lambda\left(15(0.4)K^{0.5}L^{-0.6}\right) = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = 10{,}000 - 15K^{0.5}L^{0.4} = 0$$

Solve for L as a function of K using the first two conditions:

$$\lambda = \frac{25}{15(0.5)K^{-0.5}L^{0.4}} = \frac{W}{15(0.4)K^{0.5}L^{-0.6}}$$

$$150K^{0.5}L^{-0.6} = 7.5K^{-0.5}L^{0.4}$$

$$20K^{0.5}K^{0.5} = WL^{0.6}L^{0.4}$$

$$20K = WL$$

$$K = \frac{WL}{20}$$

Substitute into the quantity constraint:

$$10{,}000 = 15K^{0.5}L^{0.4}$$

$$10{,}000 = 15\left(\frac{WL}{20}\right)^{0.5}L^{0.4}$$

$$15\left(\frac{W}{20}\right)^{0.5}L^{0.9} = 10{,}000$$

$$L^{0.9} \approx 2{,}981.42W^{-0.5}$$

$$L \approx 7{,}252.21W^{-0.56}$$

h. Take the derivative of labor demand with respect to the wage rate:

$$\frac{\partial L(W)}{\partial W} = (-0.56)7{,}252.21W^{-1.56} = -4{,}061.24W^{-1.56}$$

which is negative, as we expect from the Law of Demand.